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International Journal of Solids and Structures 43 (2006) 5690–5712
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
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Energy theorems and the *J*-integral in dipolar gradient elasticity

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Received 9 May 2005
Available online 4 October 2005

Abstract

Within the framework of Mindlin's dipolar gradient elasticity, general energy theorems are proved in this work. These are the theorem of minimum potential energy, the theorem of minimum complementary potential energy, a variational principle analogous to that of the Hellinger–Reissner principle in classical theory, two theorems analogous to those of Castigliano and Engesser in classical theory, a uniqueness theorem of the Kirchhoff–Neumann type, and a reciprocal theorem. These results can be of importance to computational methods for analyzing practical problems. In addition, the *J*-integral of fracture mechanics is derived within the same framework. The new form of the *J*-integral is identified with the energy release rate at the tip of a growing crack and its path-independence is proved.

The theory of dipolar gradient elasticity derives from considerations of microstructure in elastic continua [Mindlin, R.D., 1964. Microstructure in linear elasticity. Arch. Rational Mech. Anal. 16, 51–78] and is appropriate to model materials with periodic structure. According to this theory, the strain-energy density assumes the form of a positive-definite function of the strain (as in classical elasticity) and the second gradient of the displacement (additional term). Specific cases of the general theory considered here are the well-known theory of couple-stress elasticity and the recently popularized theory of strain-gradient elasticity. The latter case is also treated in the present study.

Keywords: Energy theorems; Variational principles; Uniqueness; Crack problems; Energy release rate; *J*-integral; Microstructure; Generalized continuum theories; Dipolar stresses; Gradient elasticity

1. Introduction

The variational principles and the related energy theorems of the classical theory of elasticity are powerful tools in deriving *numerical* solutions based on the finite element and the boundary element methods (see, e.g., Hughes, 1987; Beskos, 1987). The equations of generalized continuum theories like *gradient elasticity* are even more complicated than those of classical elasticity. Therefore, the need for analogous energy principles and theorems within the new theories is even more pronounced than in the classical theory. Here, we establish all basic energy principles and theorems for the theory of dipolar gradient elasticity based on forms I and

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.08.009

II of the general Mindlin's (1964) theory. Also, the J-integral of fracture mechanics is derived within the same framework and is identified with the energy release rate at the tip of a growing crack.

The theory of dipolar gradient elasticity (or, simply, gradient elasticity) was introduced by Mindlin (1964) in an effort to model the mechanical behavior of solids with *microstructure*. The basic concept of this generalized continuum theory lies in the consideration of a medium containing elements or particles (called macro-media), which are in themselves deformable media. This behavior can easily be realized if such a macro-particle is viewed as a collection of smaller sub-particles (called micro-media). In this way, each particle of the continuum is endowed with an *internal* displacement field, which can be expanded as a power series in internal coordinate variables. Within the above context, the lowest-order theory (dipolar or grade-two theory) is the one obtained by retaining only the first (linear) term of the foregoing series. Also, because of the inherent dependence of the strain energy on gradients of certain fields—like the displacement gradient (form I in Mindlin, 1964), the strain (form II) or the rotation (couple-stress case)—the new material constants imply the presence of characteristic lengths in the material behavior. These lengths can be related with the size of microstructure. In this way, *size effects* can be incorporated in the stress analysis in a manner that classical theories cannot afford. Continua for which such an analysis can be useful are periodic material structures like those, e.g., of crystal lattices, crystallites of a polycrystal or grains of a granular material.

Historically, ideas underlying generalized continuum theories were advanced already in the 19th century by Cauchy (1851), Voigt (1887), and the brothers Cosserat and Cosserat (1909), but the subject was generalized and reached maturity only in the 1960s with the works of Toupin (1962), Mindlin (1964), and Green and Rivlin (1964). In a brief literature review now, it should be noticed that the Mindlin theory and related ideas (Bleustein, 1967; Mindlin and Eshel, 1968) enjoyed an early and successful application (see, e.g., Weitsman, 1966; Day and Weitsman, 1966; Bleustein, 1966; Herrmann and Achenbach, 1968; Dillon and Kratochvil, 1970; Oden et al., 1970; Germain, 1973; Tiersten and Bleustein, 1975). More recently, this approach and related extensions have also been employed to analyze various problems in, among other areas, wave propagation (Vardoulakis and Georgiadis, 1997; Georgiadis et al., 2000; Georgiadis and Velgaki, 2003; Georgiadis et al., 2004), fracture (Zhang et al., 1998; Chen et al., 1998; Chen et al., 1999; Shi et al., 2000; Georgiadis, 2003; Radi and Gei, 2004; Grentzelou and Georgiadis, 2005), mechanics of defects (Lubarda and Markenskoff, 2003), and plasticity (see, e.g., Fleck et al., 1994; Vardoulakis and Sulem, 1995; Wei and Hutchinson, 1997; Begley and Hutchinson, 1998; Fleck and Hutchinson, 1998; Gao et al., 1999; Huang et al., 2000; Huang et al., 2004). In particular, recent work by Gao et al. (1999) and Huang et al. (2000, 2004) introduced a mechanism-based strain gradient theory of plasticity that is established from the Taylor dislocation model. In addition, efficient numerical techniques (see, e.g., Shu et al., 1999; Amanatidou and Aravas, 2002; Engel et al., 2002; Tsepoura et al., 2002; Giannakopoulos et al., submitted for publication; Tsamasphyros et al., 2005) have been developed to deal with some problems analyzed with the Toupin–Mindlin approach. Based on the existing results, it is concluded that the Toupin–Mindlin theory does extend the range of applicability of the 'continuum' concept in an effort to bridge the gap between classical continuum theories and atomic-lattice theories. Finally, another interesting feature of this theory is the emergence, in some cases, of *boundary layer* effects that can capture corresponding phenomena (see, e.g., Shi et al., 2000; Georgiadis, 2003; Georgiadis et al., 2004). Of course, such boundary layer effects constitute a challenge for numerical methods.

Regarding now appropriate length scales for strain gradient theories, as noted by Zhang et al. (1998), although strain gradient effects are associated with geometrically necessary dislocations in plasticity, they may also be important for the *elastic* range in microstructured materials. Indeed, Chen et al. (1998) developed a continuum model for cellular materials and found out that the continuum description of these materials obey an elasticity micropolar theory (i.e. a theory with strain gradient effects). In the latter study, the intrinsic material length was naturally identified with the cell size. Other examples of the size effect in elastically deformed solids include propagation of waves with small wavelengths in layered media (Herrmann and Achenbach, 1968), bending of a polycrystalline aluminum beam (Kakunai et al., 1985), and buckling of elastic fibers in composites (Fleck and Shu, 1995). Generally, theories with strain gradient effects are intended to model situations where the intrinsic material lengths are of the order of 0.1–10 μm (see, e.g., Shi et al., 2000). Since the strengthening effects arising from strain gradients become important when these gradients are large enough, these effects will be significant when the material is deformed in *very small* volumes, such as in the immediate vicinity of crack tips, notches, small holes and inclusions, and micrometer indentations.

Also, in wave propagation dealing with electronic-device applications, surface-wave frequencies on the order of GHz are often used and therefore wavelengths on the micron order appear (see, e.g., White, 1970; Farnell, 1978). In such situations, *dispersion phenomena* at high frequencies can only be explained on the basis of strain gradient theories (Georgiadis and Velgaki, 2003; Georgiadis et al., 2004). In addition, the latter studies also provide estimates for a single microstructural parameter (i.e. an additional material parameter to the standard Lame constants λ and μ) employed in some simple material models (like the couple-stress elasticity and the gradient elasticity of form II in Mindlin's theory), which lie within the context discussed here. Indeed, by considering that the material is composed wholly of unit cells having the form of cubes with edges of size $2h$ and by comparing the forms of dispersion curves of *Rayleigh waves* obtained by the Toupin–Mindlin approaches (in the case of either a couple-stress model or a 'pure' gradient model with no couple-stresses) with the ones obtained by the atomic-lattice analysis of Gazis et al. (1960), it can be estimated that the so-called couple-stress modulus η is of the order of $0.1\mu h^2$ (Georgiadis and Velgaki, 2003), whereas the so-called gradient coefficient c is of the order of $(0.1h)^2$ (Georgiadis et al., 2004). Finally, we should mention the work by Chang et al. (2003), who provide estimates for the microstructural constants in granular materials modeled as strain-gradient continua.

In the present study, the most common version of Mindlin's theory is employed, i.e. the so-called micro-homogeneous case (see Section 10 in Mindlin, 1964). According to this, on the one hand, each material particle has three degrees of freedom (the displacement components—just as in the classical theories) and the micro-density does not differ from the macro-density, but, on the other hand, the Euler–Cauchy principle (see, e.g., Fung, 1965; Jaunzemis, 1967) assumes a form with *non-vanishing* couple-stress vector and the strain-energy density depends not only upon the strain (as in standard elasticity) but also upon the *second gradient* of the displacement. This case is different from the general Cosserat (or micropolar) theory that takes material particles with six independent degrees of freedom (three displacement components and three rotation components, the latter involving rotation of a micro-medium w.r.t. its surrounding medium) and, as explained in Section 2 below, includes as important special cases the strain-gradient elasticity (form II in Mindlin, 1964) and the couple-stress elasticity (special case of form III in Mindlin, 1964).

Regarding now previous work on energy principles and theorems within the framework of gradient theories, we should mention that some *particular* results exist scattered in the following works that mainly deal with numerical methods. More specifically, Smyshlyaev and Fleck (1995, 1996) presented minimum energy principles applied to composites and polycrystals, Shu et al. (1999) presented a weak variational formulation for Finite Element analysis, Amanatidou and Aravas (2002) presented the principle of virtual work in the case of form III of Mindlin's theory, Polyzos et al. (2003) presented a reciprocal theorem in the case of form II of Mindlin's theory and only for isotropic material response, and Giannakopoulos et al. (submitted for publication) presented a reciprocal theorem and a theorem of Castigliano's type in the case of form II of Mindlin's theory. Giannakopoulos et al. (submitted for publication) kindly acknowledged earlier unpublished work by the present authors on similar theorems. In the present paper, we aim at a systematic derivation of *all* basic energy theorems for the most general case of Mindlin's theory, namely for form I. Finally, after treating here the general case of dipolar gradient theory (which involves the entire field of displacement gradient), we also deal with the special case of strain-gradient theory (form II).

2. Fundamentals of dipolar gradient elasticity

The general case of dipolar gradient theory for a 3D continuum is best described, assuming small strains and displacements, by the following form of the first law of thermodynamics with respect to a Cartesian rectangular coordinate system $Ox_1x_2x_3$ (indicial notation and the summation convention will be used throughout)

$$\rho \dot{\mathsf{E}} = \tau_{pq}\dot{\varepsilon}_{pq} + m_{rpq}\partial_r\partial_p\dot{u}_q, \tag{1}$$

where ρ is the mass density of the continuum, E is the internal energy per unit mass, u_q is the displacement vector, $\varepsilon_{pq} = (1/2)(\partial_p u_q + \partial_q u_p)$ is the linear strain tensor, $\tau_{pq} = \tau_{qp}$ is the monopolar (or Cauchy in the nomenclature of Mindlin, 1964) stress tensor, $m_{rpq} = m_{prq}$ is the dipolar (or double) stress tensor (a third-rank tensor), $\partial_p(\ ) \equiv \partial(\ )/\partial x_p$, a superposed dot denotes time derivative, and the Latin indices span the range $(1,2,3)$. Clearly, the above form of the first law of thermodynamics can be viewed as a more accurate description of the material response than that provided by the standard theory (case of $\rho\dot{\mathsf{E}} = \tau_{pq}\dot{\varepsilon}_{pq}$), if one thinks of a series expansion for $\rho\dot{\mathsf{E}}$ containing higher-order gradients of the displacement gradient (or even of its symmetrical part, the strain). For instance, the additional terms may become significant in the vicinity of stress-concentration points where the displacement gradient undergoes steep variations.

The dipolar stress tensor now follows from the notion of *dipolar forces*, which are anti-parallel forces acting between the micro-media contained in the continuum with microstructure (see Fig. 1). As explained by Green and Rivlin (1964) and Jaunzemis (1967), the notion of multipolar forces arises from a series expansion of the mechanical power M containing higher-order velocity gradients, i.e. $\mathsf{M} = F_q\dot{u}_q + F_{pq}(\partial_p\dot{u}_q) + F_{rpq}(\partial_r\partial_p\dot{u}_q) + \cdots$, where F_q are the usual (monopolar) forces of classical continuum mechanics and $(F_{pq}, F_{rpq}, \ldots)$ are the multipolar (dipolar, quadrupolar, etc.) forces within the framework of generalized continuum mechanics.

In this way, the resultant force on an ensemble of sub-particles can be viewed as being decomposed into *external* and *internal* forces, the latter ones being self-equilibrating. However, these self-equilibrating forces produce *non-vanishing* stresses, the multipolar stresses. This means that an element along a section or at the surface may transmit, besides the usual force vector, a *couple* vector as well (i.e. the Euler–Cauchy stress principle is augmented to include additional couple-tractions). Regarding the notation of the dipolar forces and stresses, the first index of the force indicates the orientation of the lever arm between the forces and the second one the orientation of the pair of forces itself. The same holds true for the last two indices of the dipolar stresses, whereas the first index denotes the orientation of the normal to the surface upon which the stress acts. Also, the dipolar forces F_{pq} have dimensions of [force][length]; their diagonal terms are double forces without moment and their off-diagonal terms are double forces with moment. In particular, the anti-symmetric part $F_{[pq]} = (1/2)\,(x_pF_q - x_qF_p)$ gives rise to couple-stresses. Finally, across a section with its outward unit normal in the positive direction, the force at the positive end of the lever arm is positive if it acts in the positive direction. 'Positive' refers to the positive sense of the coordinate axis parallel to the lever arm or force.

Next, compatible with (1) is the following form of the *strain-energy* density W stored in the continuum:

$$W \equiv W(\varepsilon_{pq}, \kappa_{rpq}), \tag{2}$$

where $\kappa_{rpq} = \partial_r\partial_p u_q = \partial_p\partial_r u_q$ is the second gradient of displacement. The rotation tensor $\omega_{pq} = (1/2)(\partial_p u_q - \partial_q u_p)$ is also recorded for future reference. The kinematical field $(\varepsilon_{pq}, \kappa_{rpq})$ is assumed to be compatible in the sense that the relations $e_{ljp}e_{mqr}\partial_j\partial_r\varepsilon_{pq} = 0$ and $e_{ljr}\partial_j\kappa_{rpq} = 0$ (with e_{ljp} being the Levi–Civita alternating symbol) are satisfied (cf. Mindlin, 1964). In what follows, we assume the existence of a *positive definite* function $W(\varepsilon_{pq}, \kappa_{rpq})$. Also, the form in (2) allows not only for a linear constitutive behavior of the material but also for a *non-linear* one. Indeed, in the present work, except for the cases of uniqueness theorem and reciprocal theorem, all other energy theorems are valid for non-linear constitutive behavior as well. From the previous definitions of kinematical variables, the properties $\varepsilon_{pq} = \varepsilon_{qp}$, $\kappa_{rpq} = \kappa_{prq}$ and $\omega_{pq} = -\omega_{qp}$ are obvious. Simpler versions of the general theory (form I in Mindlin, 1964) can be derived by identifying κ_{rpq} with either the strain gradient (strain-gradient theory: $\kappa_{rpq} = \partial_r\varepsilon_{pq}$—form II in Mindlin, 1964) or the rotation gradient (couple-stress theory: $\kappa_{rpq} = \partial_r\omega_{pq}$—special case of form III in Mindlin, 1964). Nevertheless, we deal here with

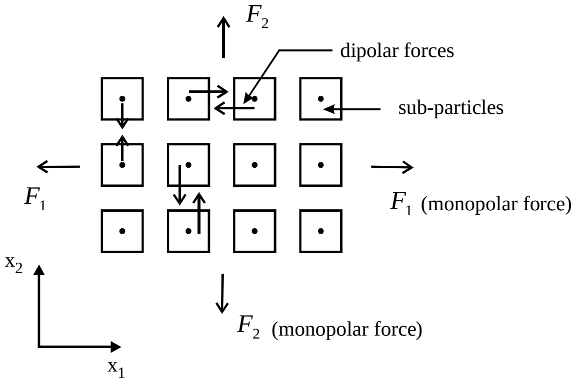

Fig. 1. A solid with microstructure: monopolar (external) and dipolar (internal) forces acting on an ensemble of sub-particles.

the general case by taking the gradient of the *entire* displacement-gradient field. In the final Section of the paper, the results for the special case of strain-gradient theory will also be presented.

Further, stresses can be defined in the standard variational manner

$$\tau_{pq} \equiv \frac{\partial W}{\partial \varepsilon_{pq}}, \quad m_{rpq} \equiv \frac{\partial W}{\partial \kappa_{rpq}}. \tag{3a, b}$$

Also, for Eq. (3) to establish a one-to-one correspondence between τ_{pq} and ε_{pq} and between m_{rpq} and κ_{rpq} (i.e. in order for (3) to be invertible being able, therefore, to provide ε_{pq} and κ_{rpq} in terms of τ_{pq} and m_{rpq}, respectively), the following conditions should prevail in view of the implicit function theorem: (i) the derivatives $(\partial^2 W/\partial\varepsilon_{pq}\partial\varepsilon_{lm})$, $(\partial^2 W/\partial\kappa_{rpq}\partial\kappa_{jlm})$ and $(\partial^2 W/\partial\varepsilon_{pq}\partial\kappa_{jlm})$ be continuous in the neighborhood of the 'point' $(\varepsilon^*_{pq}, \kappa^*_{rpq})$, and (ii) the determinants $|\partial^2 W/\partial\varepsilon_{pq}\partial\varepsilon_{lm}|$, $|\partial^2 W/\partial\kappa_{rpq}\partial\kappa_{jlm}|$ and $|\partial^2 W/\partial\varepsilon_{pq}\partial\kappa_{jlm}|$ be different than zero in the neighborhood of the 'point' $(\varepsilon^*_{pq}, \kappa^*_{rpq})$.

In the case now of a *linear* constitutive behavior, the strain-energy density takes the following general quadratic form:

$$W = (1/2)c_{pqlm}\varepsilon_{pq}\varepsilon_{lm} + (1/2)d_{rpqjlm}\kappa_{rpq}\kappa_{jlm} + f_{rpqlm}\kappa_{rpq}\varepsilon_{lm}, \tag{4}$$

where c_{pqlm}, d_{rpqjlm} and f_{rpqlm} are tensors of the material constants. The number of independent components of the tensors c_{pqlm} and d_{rpqjlm} (which are of even rank) can be reduced to yield isotropic behavior, whereas the tensor f_{rpqlm} (being of odd rank) inevitably results in some type of anisotropic behavior (i.e. in preferred directions in the material response). In other words, when isotropic material behavior is to be considered, it should definitely be set $f_{rpqlm} = 0$. In the general case, $(c_{pqlm}, d_{rpqjlm}, f_{rpqlm})$ can be considered as continuously differentiable functions of position (case of non-homogeneous behavior). On the other hand, the positive definiteness of W sets the usual restrictions on the range of values of the material constants. Inequalities of this type are given, e.g., in Georgiadis et al. (2004) for the isotropic strain-gradient case. In addition, due to symmetries of the stresses and the kinematical variables and, also, due to the conditions stated immediately below Eq. (3), the following symmetries prevail too:

$$c_{pqlm} = c_{lmpq} = c_{qplm} = c_{pqml}, \quad d_{rpqjlm} = d_{jlmrpq} = d_{prqjlm} = d_{rpqljm}, \quad f_{rpqlm} = f_{rpqml} = f_{prqlm}. \tag{5a, b, c}$$

According to the definitions in (3), Eq. (4) provides the following general linear constitutive relations:

$$\tau_{pq} = c_{pqlm}\varepsilon_{lm} + f_{pqjlm}\kappa_{jlm}, \quad m_{rpq} = f_{rpqlm}\varepsilon_{lm} + d_{rpqjlm}\kappa_{jlm}. \tag{6a, b}$$

Finally, as an example of linear constitutive relations, we record the case of an *isotropic* material (Mindlin, 1964)

$$\tau_{pq} = \lambda\delta_{pq}\varepsilon_{jj} + 2\mu\varepsilon_{pq}, \tag{7a}$$

$$m_{rpq} = \frac{1}{2}d_1(\kappa_{jjr}\delta_{pq} + 2\kappa_{qjj}\delta_{rp} + \kappa_{jjp}\delta_{qr}) + d_2(\kappa_{rjj}\delta_{pq} + \kappa_{pjj}\delta_{rq}) + 2d_3\kappa_{jjq}\delta_{rp} + 2d_4\kappa_{rpq} + d_5(\kappa_{qpr} + \kappa_{qrp}), \tag{7b}$$

where λ and μ are the standard Lamé constants, and d_α $(\alpha = 1, \ldots, 5)$ are the additional material constants.

Next, the equations of equilibrium (global equilibrium) and the *traction* boundary conditions along a smooth boundary (local equilibrium) can be obtained from variational considerations (Mindlin, 1964). In particular, the issue of traction boundary conditions and their nature was elucidated by Bleustein (1967) in an important but not so widely known paper. These equations read (the first is the equation of equilibrium and the other two are the traction boundary conditions)

$$\partial_p(\tau_{pq} - \partial_r m_{rpq}) + f_q = 0 \quad \text{in } V, \tag{8}$$

$$n_p(\tau_{pq} - \partial_r m_{rpq}) - D_p(n_r m_{rpq}) + (D_j n_j)n_r n_p m_{rpq} = P_q \quad \text{on } bdy, \tag{9}$$

$$n_r n_p m_{rpq} = R_q \quad \text{on } bdy, \tag{10}$$

where V is the region (open set) occupied by the body, bdy denotes any boundary along a section inside the body or along the surface of it, f_q is the monopolar body force per unit volume, $D_p(\ ) \equiv \partial_p(\ ) - n_p D(\ )$ is the surface gradient operator, $D(\ ) \equiv n_r\partial_r(\ )$ is the normal gradient operator, n_p is the outward unit normal to the boundary, $P_q \equiv t^{(n)}_q + (D_r n_r)n_p T^{(n)}_{pq} - D_p T^{(n)}_{pq}$ is the auxiliary force traction, $R_q \equiv n_p T^{(n)}_{pq}$ is the auxiliary

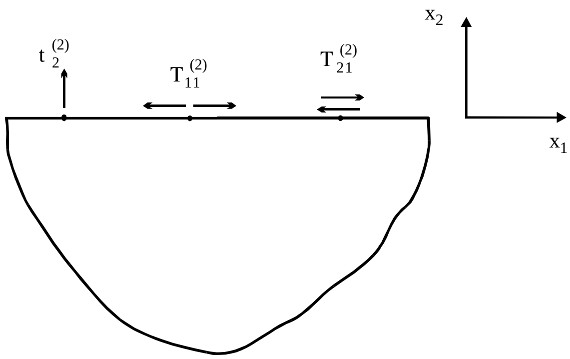

Fig. 2. Positively oriented true monopolar and dipolar tractions on the surface of a half-space.

double force traction, $t_q^{(n)}$ is the *true* force surface traction, and $T_{pq}^{(n)}$ is the *true* double force surface traction. Examples of the latter tractions along the surface of a 2D half-space are given in Fig. 2. Also, concrete boundary value problems can be found in, e.g., Georgiadis (2003), Georgiadis et al. (2004), and Grentzelou and Georgiadis (2005). Finally, let S_σ be the portion of the surface S of the body on which external tractions are prescribed.

The other type of boundary conditions (i.e. the *kinematical* boundary conditions) is stated now and, in fact, these boundary conditions (along with the traction boundary conditions) will be verified here in the context of the Principle of Complementary Virtual Work and the principle that is analogous to the one of Hellinger–Reissner in classical theory. These boundary conditions are

$$u_q: \quad \text{given on } S_u, \tag{11}$$

$$D(u_q): \quad \text{given on } S_u, \tag{12}$$

where S_u is the portion of the surface S of the body on which both displacements and their normal derivatives are prescribed. Of course, $S_\sigma \cup S_u = S$ and $S_\sigma \cap S_u = \emptyset$ hold true.

Finally, we have two notices regarding: (i) dipolar body forces, and (ii) the case of possible edges formed by the intersection of two portions of the closed boundary surface S. First, a dipolar body force field is omitted in (8) and (9) since this case is a rather unrealistic possibility. This absence of double body forces can also be quoted in Mindlin's (1964) form I and, also, in Mindlin and Eshel (1968). Second, if an edge E is formed by the intersection of two portions, say, S_1 and S_2 of S, then $\|\tilde{n}_p n_r m_{rpq}\| = \|\tilde{n}_p T_{pq}^{(n)}\|$ on E, where $\tilde{n}_q = e_{rpq} s_r n_p$ with e_{rpq} being the Levi–Civita alternating symbol and s_r being the unit tangent vector to the curve E, and the brackets $\|\,\|$ denote that the enclosed quantity is the difference between the values on S_1 and S_2 (Mindlin, 1964; Bleustein, 1967). In what follows, we omit generally this case with the understanding that, when such a situation arises, appropriate terms stemming from it can be added to our results. As an example, this will be indicated for the Principle of Virtual Work below.

3. Preliminaries—principle of virtual work

In this section, we will arrive at the Principle of Virtual Work starting from Eqs. (8)–(10). This is an alternative to the way advanced by Mindlin (1964) to verify that the Principle of Virtual Work is a necessary and sufficient condition to satisfy equilibrium. We finally end up with an integral equation (weak form) that could be solved for the unknowns (τ_{pq}, m_{rpq}) in any particular problem.

Let a body that occupies the region V enclosed by a (piecewise smooth) surface S be in equilibrium. The body is under the action of prescribed body forces f_q, prescribed single and double surface forces (P_q^*, R_q^*) on the portion S_σ of the boundary, and prescribed displacements and their normal derivatives (u_q^*, Du_q^*) on the portion S_u of the boundary. Then, provided that Eqs. (8)–(10) hold, we construct an *integral expression* identically equal to zero for a virtual field $(\delta u_q, D(\delta u_q))$, which is kinematically admissible (i.e. sufficiently differentiable, compatible with the assumption of infinitesimal strains, and such that $\delta u_q = 0$ and $D(\delta u_q) = 0$ on S_u) but otherwise arbitrary (see Fig. 3)

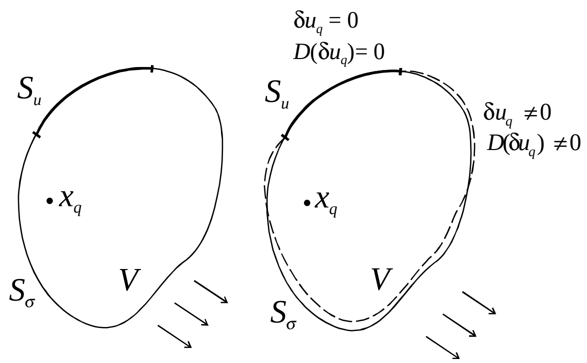

Fig. 3. Virtual deformation of the body.

$$\int_V \left[\partial_p\left(\tau_{pq}-\partial_r m_{rpq}\right)+f_q\right]\delta u_q\,\mathrm{d}V-\int_{S_\sigma}\left[n_p\left(\tau_{pq}-\partial_r m_{rpq}\right)-D_p\left(n_r m_{rpq}\right)+(D_l n_l)n_r n_p m_{rpq}-P_q^*\right]\delta u_q\,\mathrm{d}S$$
$$-\int_{S_\sigma}(n_r n_p m_{rpq}-R_q^*)D(\delta u_q)\mathrm{d}S=0, \tag{13}$$

where the operator δ( ) denotes weak variations (see, e.g., Gelfand and Fomin, 1963; Dym and Shames, 1973). Obviously, in the general 3D case, integrals over V are volume integrals, whereas integrals over S (or its portions) are surface integrals.

Now, employing the Green–Gauss theorem along with standard tensor properties and the definitions of kinematical quantities, the first integral in the LHS of (13) becomes

$$\int_V \left[\partial_p\left(\tau_{pq}-\partial_r m_{rpq}\right)+f_q\right]\delta u_q\,\mathrm{d}V$$
$$=\int_V \partial_p\left[\left(\tau_{pq}-\partial_r m_{rpq}\right)\delta u_q\right]\mathrm{d}V-\int_V\left(\tau_{pq}-\partial_r m_{rpq}\right)\delta(\partial_p u_q)\mathrm{d}V+\int_V f_q\delta u_q\,\mathrm{d}V$$
$$=\int_S n_p\left(\tau_{pq}-\partial_r m_{rpq}\right)\delta u_q\,\mathrm{d}S+\int_S n_r m_{rpq}\partial_p(\delta u_q)\mathrm{d}S-\int_V \tau_{pq}\delta\varepsilon_{pq}\,\mathrm{d}V-\int_V m_{rpq}\delta\kappa_{rpq}\,\mathrm{d}V+\int_V f_q\delta u_q\,\mathrm{d}V. \tag{14}$$

Further, by defining the internal virtual work as

$$\delta U\equiv\int_V\left(\tau_{pq}\delta\varepsilon_{pq}+m_{rpq}\delta\kappa_{rpq}\right)\mathrm{d}V, \tag{15}$$

Eq. (14) takes the form

$$\int_V \left[\partial_p\left(\tau_{pq}-\partial_r m_{rpq}\right)+f_q\right]\delta u_q\,\mathrm{d}V=\int_S n_p\left(\tau_{pq}-\partial_r m_{rpq}\right)\delta u_q\,\mathrm{d}S+\int_S n_r m_{rpq}\partial_p\left(\delta u_q\right)\mathrm{d}S+\int_V f_q\delta u_q\,\mathrm{d}V-\delta U. \tag{16}$$

Next, in view of (16) and the fact that $\delta u_q=0$ and $D(\delta u_q)=0$ on S_u, the identity in (13) is written as

$$\int_{S_u} n_p\left(\tau_{pq}-\partial_r m_{rpq}\right)\delta u_q\,\mathrm{d}S+\int_{S_\sigma}\left[D_p\left(n_r m_{rpq}\right)-(D_l n_l)n_r n_p m_{rpq}\right]\delta u_q\,\mathrm{d}S+\int_S n_r m_{rpq}\partial_p(\delta u_q)\mathrm{d}S$$
$$+\int_{S_\sigma}\left[P_q\delta u_q+R_q D(\delta u_q)\right]\mathrm{d}S+\int_V f_q\delta u_q\,\mathrm{d}V-\delta U-\int_{S_\sigma} n_r n_p m_{rpq}D(\delta u_q)\mathrm{d}S=0. \tag{17}$$

In addition, by taking into account the definition of the work of external forces

$$W^{\text{ext}} \equiv \int_V f_q u_q \, \mathrm{d}V + \int_{S_\sigma} [P_q u_q + R_q D(u_q)] \mathrm{d}S, \tag{18}$$

and its first variation (the external forces are kept *constant* during the virtual deformation—i.e. a loading of non-follower type is assumed)

$$\delta W^{\text{ext}} = \int_V f_q \delta u_q \, \mathrm{d}V + \int_S \left[P_q \delta u_q + R_q D(\delta u_q)\right] \mathrm{d}S \tag{19}$$

Eq. (17) takes the form

$$\delta W^{\text{ext}} - \delta U + \int_S n_r m_{rpq} \partial_p(\delta u_q) \mathrm{d}S - \int_S n_r n_p m_{rpq} D\big(\delta u_q\big) \mathrm{d}S$$
$$+ \int_S \left[D_p\big(n_r m_{rpq} \delta u_q\big) - (D_l n_l) n_r n_p m_{rpq} \delta u_q - n_r m_{rpq} D_p\big(\delta u_q\big)\right] \mathrm{d}S. \tag{20}$$

At this step, use will be made of the following relation:

$$n_k e_{krq} \partial_r (e_{lqp} n_p n_j T_{abc\ldots}) = D_l(n_j T_{abc\ldots}) - n_l n_j T_{abc\ldots}(D_k n_k), \tag{21}$$

which can be proved to hold true for a Cartesian tensor $T_{abc\ldots}$ of any rank. The proof is straightforward and, thus, is omitted. Considering now the case $T_{rp} \equiv m_{rpq}\delta u_q$ in (21), we have

$$n_s e_{sjk} \partial_j (e_{pkl} n_l n_r m_{rpq} \delta u_q) = D_p(n_p m_{rpq} \delta u_q) - (D_l n_l) n_r n_p m_{rpq} \delta u_q, \tag{22}$$

which when integrated over the boundary S provides

$$\int_S D_p(n_r m_{rpq} \delta u_q) \mathrm{d}S = \int_S (D_l n_l) n_r n_p m_{rpq} \delta u_q \, \mathrm{d}S, \tag{23}$$

since applying the Green–Gauss theorem to the surface integral resulting from the term in the LHS of (22) gives zero.

Then, in view of (23), we write (20) under the form

$$\delta W^{\text{ext}} - \delta U + \int_S n_r m_{rpq} \partial_p\big(\delta u_q\big) \mathrm{d}S - \int_S n_r m_{rpq} D_p\big(\delta u_q\big) \mathrm{d}S - \int_S n_r n_p m_{rpq} D\big(\delta u_q\big) \mathrm{d}S = 0, \tag{24}$$

and finally, by recalling that $\partial_p(\ ) = D_p(\ ) + n_p D(\ )$, we succeed in proving

$$\delta W^{\text{ext}} = \delta U, \tag{25a}$$

or, more explicitly,

$$\int_V f_q \delta u_q \, \mathrm{d}V + \int_S \left[P_q \delta u_q + R_q D\big(\delta u_q\big)\right] \mathrm{d}S = \int_V \big(\tau_{pq} \delta \varepsilon_{pq} + m_{rpq} \delta \kappa_{rpq}\big) \mathrm{d}V. \tag{25b}$$

Eq. (25b) is the mathematical expression of the Principle of Virtual Work.

We conclude this section with the following remarks: (i) Eq. (25b) is *independent* of any constitutive law since in the previous procedure the virtual field $(\delta u_q, D(\delta u_q))$ was not related to the stress field (τ_{pq}, m_{rpq}). If, however, a non-linear elasticity law is assumed, then the internal virtual work δU is identified with the *first variation* of the strain energy stored in the body $\int_V W(\varepsilon_{pq},\ \kappa_{rpq}) \mathrm{d}V$, where $W(\varepsilon_{pq}, \kappa_{rpq}) \equiv (\int_0^{\varepsilon_{pq}} \tau_{pq} \mathrm{d}\varepsilon_{pq} + \int_0^{\kappa_{rpq}} m_{rpq} \mathrm{d}\kappa_{rpq})$ in view of (3). (ii) Eq. (25b) applies only for infinitesimal deformation. (iii) Eq. (25b) is a *necessary* condition for the global and local equilibrium of the body. It is also a *sufficient* condition, for if (25b) holds, then retracting the argument in the reversed direction and considering that the field $(\delta u_q, D(\delta u_q))$ is arbitrary, one arrives at Eqs. (8)–(10). (iv) In the case of an edge E formed by the intersection of two smooth portions of the boundary S, the term $\int_E \|\tilde{n}_p T_{pq}^{(n)}\| \delta u_q \, \mathrm{d}s$ (relative symbols have been defined at the end of Section 2) should be added to the terms in the LHS of (25b).

4. Theorem of minimum potential energy

With the help of the Principle of Virtual Work, we will prove here the Theorem of Minimum Potential Energy for solids governed by non-linear gradient elasticity. First, it is shown that the so-called potential energy (a functional which will be defined below) exhibits an extremum for the actual deformation of the body. This deformation (true solution for the displacement field in a particular problem) satisfies the equilibrium equations and the elastic constitutive law, in addition to the conditions satisfied by all kinematically admissible fields. In this way, the stationary value of the potential energy *singles out* the true solution from the infinity of kinematically-admissible candidate solutions. Finally, it will be proved that the aforementioned extremum is actually a *minimum*. As in the case of classical elasticity, the present theorem may serve for approximate procedures of the Rayleigh–Ritz type.

The (total) potential energy of the system is defined as

$$\Pi \equiv \int_V W(\varepsilon_{pq}, \kappa_{rpq}) \mathrm{d}V - \int_V f_q u_q \, \mathrm{d}V - \int_{S_\sigma} [P_q u_q + R_q D(u_q)] \mathrm{d}S \tag{26}$$

for the true solution of the problem and for a non-follower type of loading. Then, it follows immediately from (19), (25) and (26) that

$$\delta\Pi = 0. \tag{27}$$

Consider now an additional field to the true displacement field of the problem. This additional field can be identified with a kinematically admissible virtual field $(\delta u_q, D(\delta u_q))$. The corresponding potential energy for this new (kinematically admissible) configuration of the body assumes the form

$$\begin{aligned}\Pi(\{u_q + \delta u_q\}, \{D(u_q) + D(\delta u_q)\}) = &\int_V [W(\{\varepsilon_{pq} + \delta\varepsilon_{pq}\}, \{\kappa_{rpq} + \delta\kappa_{rpq}\}) - f_q(u_q + \delta u_q)] \mathrm{d}V \\ &- \int_{S_\sigma} [P_q(u_q + \delta u_q) + R_q D(u_q + \delta u_q)] \mathrm{d}S.\end{aligned} \tag{28}$$

Further, the difference of the potential energies between the two configurations is defined as

$$\begin{aligned}\Delta\Pi &\equiv \Pi(\{u_q + \delta u_q\}, \{D(u_q) + D(\delta u_q)\}) - \Pi(u_q, D(u_q)) \\ &= \int_V [W(\{\varepsilon_{pq} + \delta\varepsilon_{pq}\}, \{\kappa_{rpq} + \delta\kappa_{rpq}\}) - W(\varepsilon_{pq}, \kappa_{rpq})] \mathrm{d}V \\ &\quad - \int_V f_q \delta u_q \, \mathrm{d}V - \int_{S_\sigma} [P_q \delta u_q + R_q D(\delta u_q)] \mathrm{d}S,\end{aligned} \tag{29}$$

so, our aim now will be to show that $\Delta\Pi > 0$.

To this end, on expanding the following term as a Taylor series

$$\begin{aligned}&W(\{\varepsilon_{pq} + \delta\varepsilon_{pq}\}, \{\kappa_{rpq} + \delta\kappa_{rpq}\}) \\ &\quad = W(\varepsilon_{pq}, \kappa_{rpq}) + \frac{\partial W}{\partial \varepsilon_{pq}} \delta\varepsilon_{pq} + \frac{\partial W}{\partial \kappa_{rpq}} \delta\kappa_{rpq} + \frac{1}{2!}\left(\frac{\partial^2 W}{\partial \varepsilon_{pq} \partial \varepsilon_{kl}} \delta\varepsilon_{pq} \delta\varepsilon_{kl} + 2\frac{\partial^2 W}{\partial \varepsilon_{pq} \partial \kappa_{jkl}} \delta\varepsilon_{pq} \delta\kappa_{jkl} + \frac{\partial^2 W}{\partial \kappa_{rpq} \partial \kappa_{jkl}} \delta\kappa_{rpq} \delta\kappa_{jkl}\right) + \cdots,\end{aligned} \tag{30}$$

we get

$$\begin{aligned}W(\{\varepsilon_{pq} + \delta\varepsilon_{pq}\}, \{\kappa_{rpq} + \delta\kappa_{rpq}\}) - W(\varepsilon_{pq}, \kappa_{spq}) &= \tau_{pq} \delta\varepsilon_{pq} + m_{rpq} \delta\kappa_{rpq} + \frac{1}{2!} \delta^2 W + \cdots \\ &= \delta W + \frac{1}{2!} \delta^2 W + \cdots,\end{aligned} \tag{31}$$

where

$$\delta^2 W = \frac{\partial^2 W}{\partial\varepsilon_{pq}\partial\varepsilon_{kl}}\delta\varepsilon_{pq}\delta\varepsilon_{kl} + 2\frac{\partial^2 W}{\partial\varepsilon_{pq}\partial\kappa_{jkl}}\delta\varepsilon_{pq}\delta\kappa_{jkl} + \frac{\partial^2 W}{\partial\kappa_{rpq}\partial\kappa_{jkl}}\delta\kappa_{rpq}\delta\kappa_{jkl} \tag{32}$$

is the second variation of the strain-energy density, whereas higher-order variations are defined analogously.

On using now (31) and the result $\delta\Pi = 0$ proved before, Eq. (29) yields

$$\Delta\Pi = \frac{1}{2!}\int_V \delta^2 W \,\mathrm{d}V + \cdots. \tag{33}$$

Finally, using the positive-definiteness property of the function $W(\varepsilon_{pq}, \kappa_{rpq})$ in (33), we conclude that $\Delta\Pi > 0$ and, indeed, that the potential energy assumes a minimum value for the true deformed configuration of the body.

5. Principle of complementary virtual work

Here, we will develop a conjugate or complementary principle to that developed in Section 3. Instead of considering real forces and fictitious displacements and displacement normal derivatives, we will investigate the work done by a system of *virtual* forces during the *actual* deformation. The resulting Principle of Complementary Virtual Work is a necessary and sufficient condition to satisfy a consistent deformed configuration. Thus, this principle has a kinematical character.

Let the following relations for a *compatible* kinematical field be satisfied in a body that occupies the region V enclosed by a (piecewise smooth) surface S

$$\varepsilon_{pq} = (1/2)(\partial_p u_q + \partial_q u_p), \tag{34}$$

$$\kappa_{rpq} = \partial_r\partial_p u_q. \tag{35}$$

In addition, the following kinematical boundary conditions are assumed to prevail along the portion S_u of the surface

$$u_q = u_q^*, \tag{36}$$

$$D(u_q) = D(u_q^*), \tag{37}$$

where $(u_q^*, D(u_q^*))$ are known.

The above kinematical field corresponds to a certain field of stresses and external forces. Further, we allow as admissible variations of the latter field only those satisfying the equilibrium equations and the traction boundary conditions, i.e. Eqs. (8)–(10). Then, the virtual fields $(\delta\tau_{pq}, \delta m_{rpq})$ and $(\delta f_q, \delta P_q, \delta R_q)$ will satisfy the following equations:

$$\partial_p(\delta\tau_{pq} - \partial_r(\delta m_{rpq})) + \delta f_q = 0 \quad \text{in } V, \tag{38}$$

$$n_p(\delta\tau_{pq} - \partial_r(\delta m_{rpq})) - D_p(n_r\delta m_{rpq}) + (D_j n_j)n_r n_p \delta m_{rpq} \equiv \delta P_q = 0 \quad \text{on } S_\sigma, \tag{39}$$

$$n_r n_p \delta m_{rpq} \equiv \delta R_q = 0 \quad \text{on } S_\sigma. \tag{40}$$

Now, provided that Eqs. (34)–(37) hold, we construct an *integral expression* identically equal to zero for the virtual fields

$$\int_V [\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)]\delta\tau_{pq}\,\mathrm{d}V + \int_V (\kappa_{rpq} - \partial_r\partial_p u_q)\delta m_{rpq}\,\mathrm{d}V + \int_{S_u}(u_q - u_q^*)\delta P_q\,\mathrm{d}S + \int_{S_u}[D(u_q) - D(u_q^*)]\delta R_q\,\mathrm{d}S = 0. \tag{41}$$

Next, by defining the internal complementary virtual work

$$\delta U_C \equiv \int_V (\varepsilon_{pq}\delta\tau_{pq} + \kappa_{rpq}\delta m_{rpq})\mathrm{d}V, \tag{42}$$

and using the Green–Gauss theorem and taking into consideration the symmetries of the tensors ε_{pq} and κ_{rpq}, the identity in (41) becomes

$$\delta U_{\mathrm{C}} - \int_S n_p u_q[\delta\tau_{pq} - \partial_r(\delta m_{rpq})]\mathrm{d}S + \int_V u_q\partial_p[\delta\tau_{pq} - \partial_r(\delta m_{rpq})]\mathrm{d}V - \int_S n_r(\partial_p u_q)\delta m_{rpq}\,\mathrm{d}S + \int_{S_u}(u_q - u_q^*)\delta P_q\,\mathrm{d}S + \int_{S_u}[D(u_q) - D(u_q^*)]\delta R_q\,\mathrm{d}S = 0. \tag{43}$$

Also, in light of (38)–(40), Eq. (43) takes the form

$$\delta U_{\mathrm{C}} - \int_{S_\sigma} u_q\delta P_q\,\mathrm{d}S - \int_S [D_p(n_r\delta m_{rpq}) - (D_j n_j)n_r n_p\delta m_{rpq}]u_q\,\mathrm{d}S - \int_S n_r(\partial_p u_q)\delta m_{rpq}\,\mathrm{d}S - \delta W_{\mathrm{c}}^{\mathrm{ext}} + \int_{S_u} D(u_q)\delta R_q\,\mathrm{d}S = 0, \tag{44}$$

where

$$\delta W_{\mathrm{C}}^{\mathrm{ext}} \equiv \int_V u_q\delta f_q\,\mathrm{d}V + \int_{S_u}\left[u_q^*\delta P_q + D(u_q^*)\delta R_q\right]\mathrm{d}S \tag{45}$$

is the complementary virtual work done by the virtual forces during the actual deformation of the body.

Finally, working on (44), we first note that the first integral vanishes because of (39) and then we invoke (23) and the definition of the surface gradient. In this way, we succeed in proving

$$\delta W_{\mathrm{C}}^{\mathrm{ext}} = \delta U_{\mathrm{C}}, \tag{46a}$$

or, more explicitly,

$$\int_V u_q\delta f_q\,\mathrm{d}V + \int_S\left[u_q\delta P_q + D(u_q)\delta R_q\right]\mathrm{d}S = \int_V(\varepsilon_{pq}\delta\tau_{pq} + \kappa_{rpq}\delta m_{rpq})\mathrm{d}V. \tag{46b}$$

Eq. (46b) is the mathematical expression of the Principle of the Complementary Virtual Work.

The following remarks are now in order: (i) Eq. (46b) is *independent* of any constitutive law since the virtual field $(\delta\tau_{pq}, \delta m_{rpq})$ was not related to the kinematical field $(\varepsilon_{pq}, \kappa_{rpq})$. If, however, a non-linear elasticity law is assumed, then the internal complementary virtual work δU_{C} is identified with the *first variation* of the complementary strain energy $\int_V W_{\mathrm{C}}(\tau_{pq}, m_{rpq})\mathrm{d}V$, where $W_{\mathrm{C}}(\tau_{pq}, m_{rpq}) \equiv \tau_{pq}\varepsilon_{pq} + m_{rpq}\kappa_{rpq} - W(\varepsilon_{pq}, \kappa_{rpq}) \equiv (\int_0^{\tau_{pq}}\varepsilon_{pq}\,\mathrm{d}\tau_{pq} + \int_0^{m_{rpq}}\kappa_{rpq}\,\mathrm{d}m_{rpq})$ is the complementary strain-energy density. (ii) Eq. (46b) applies only for infinitesimal deformation. (iii) Eq. (46b) is a *necessary* condition for the satisfaction of all basic kinematical relations of the dipolar theory (cf. Eqs. (34) and (35)) and the kinematical boundary conditions (cf. Eqs. (36) and (37)). It is also a *sufficient* condition for simply-connected domains.

6. Theorem of minimum complementary potential energy

With the help of the Principle of Complementary Virtual Work, we will prove here the Theorem of Minimum Complementary Potential Energy for solids governed by non-linear gradient elasticity. First, it is shown that the so-called complementary potential energy (a functional which will be defined below) exhibits an extremum for the solution characterizing the true stress state of the body. This state (true solution for the stress field in a particular problem) satisfies all kinematical relations and boundary conditions and the elastic constitutive law, in addition to the global and local equilibrium equations satisfied by all admissible stress fields. In this way, the stationary value of the complementary potential energy *singles out* the true solution from the infinity of equilibrium candidate solutions. Finally, it will be proved that the aforementioned extremum is actually a *minimum*.

For the true solution of the problem and for a conservative system of loading, the (total) complementary potential energy of the system is defined as

$$\Pi_{\mathrm{C}} \equiv \int_V W_{\mathrm{C}}(\tau_{pq}, m_{rpq})\mathrm{d}V - \int_V u_q f_q\,\mathrm{d}V - \int_{S_u}\left[u_q P_q + D(u_q)R_q\right]\mathrm{d}S, \tag{47}$$

where

$$W_{\mathrm{C}}(\tau_{pq}, m_{rpq}) \equiv \tau_{pq}\varepsilon_{pq} + m_{rpq}\kappa_{rpq} - W(\varepsilon_{pq}, \kappa_{rpq}), \tag{48}$$

while it can be checked that $(\partial W_{\mathrm{C}}/\partial \tau_{pq}) = \varepsilon_{pq}$ and $(\partial W_{\mathrm{C}}/\partial m_{rpq}) = \kappa_{rpq}$. Also, in the special case of linear constitutive behavior (cf. Eq. (4)), it can be easily shown that $W_{\mathrm{C}} = W$.

Then, it readily follows from (45)–(47) that

$$\delta \Pi_{\mathrm{C}} = 0. \tag{49}$$

Consider now an additional field to the true stress field of the problem. This additional field can be identified with the virtual field $(\delta\tau_{pq}, \delta m_{rpq})$ of Section 5. The latter field satisfies the equilibrium equations and the traction boundary conditions, i.e. Eqs. (8)–(10). The corresponding complementary potential energy for this new (statically admissible) state of the body assumes the form

$$\begin{aligned}\Pi_{\mathrm{C}}(\{\tau_{pq}+\delta\tau_{pq}\},\{m_{rpq}+\delta m_{rpq}\}) = \int_V \left[W_{\mathrm{C}}(\{\tau_{pq}+\delta\tau_{pq}\},\{m_{rpq}+\delta m_{rpq}\}) - u_q(f_q+\delta f_q)\right]\mathrm{d}V \\ - \int_{S_u}\left[u_q(P_q+\delta P_q) + D(u_q)(R_q+\delta R_q)\right]\mathrm{d}S.\end{aligned} \tag{50}$$

In addition, the difference of the complementary potential energies between the two states is defined as

$$\begin{aligned}\Delta\Pi_{\mathrm{C}} &\equiv \Pi_{\mathrm{C}}(\{\tau_{pq}+\delta\tau_{pq}\},\{m_{spq}+\delta m_{spq}\}) - \Pi_{\mathrm{C}}(\tau_{pq}, m_{spq}) \\ &= \int_V \left[W_{\mathrm{C}}(\{\tau_{pq}+\delta\tau_{pq}\},\{m_{rpq}+\delta m_{rpq}\}) - W_{\mathrm{C}}(\tau_{pq}, m_{rpq})\right]\mathrm{d}V - \int_V u_q\delta f_q\,\mathrm{d}V \\ &\quad - \int_{S_u}\left[u_q\delta P_q + D(u_q)\delta R_q\right]\mathrm{d}S,\end{aligned} \tag{51}$$

and, therefore, we would like now to show that $\Delta\Pi_{\mathrm{C}} > 0$.

To this end, we expand the term $W_{\mathrm{C}}(\{\tau_{pq}+\delta\tau_{pq}\},\{m_{rpq}+\delta m_{rpq}\})$ as a Taylor series and get

$$\begin{aligned}W_{\mathrm{C}}(\{\tau_{pq}+\delta\tau_{pq}\},\{m_{rpq}+\delta m_{rpq}\}) - W_{\mathrm{C}}(\tau_{pq}, m_{rpq}) &= \varepsilon_{pq}\delta\tau_{pq} + \kappa_{rpq}\delta m_{rpq} + \frac{1}{2!}\delta^2 W_{\mathrm{C}} + \cdots \\ &= \delta W_{\mathrm{C}} + \frac{1}{2!}\delta^2 W_{\mathrm{C}} + \cdots,\end{aligned} \tag{52}$$

where

$$\delta^2 W_{\mathrm{C}} = \frac{\partial^2 W_{\mathrm{C}}}{\partial\tau_{pq}\partial\tau_{kl}}\delta\tau_{pq}\delta\tau_{kl} + 2\frac{\partial^2 W_{\mathrm{C}}}{\partial\tau_{pq}\partial m_{jkl}}\delta\tau_{pq}\delta m_{jkl} + \frac{\partial^2 W_{\mathrm{C}}}{\partial m_{rpq}\partial m_{jkl}}\delta m_{rpq}\delta m_{jkl} \tag{53}$$

is the second variation of the complementary strain-energy density, and higher-order variations are defined in the same way.

On using now (52) and the result $\delta\Pi_{\mathrm{C}} = 0$ proved before, Eq. (51) yields

$$\Delta\Pi_{\mathrm{C}} = \frac{1}{2!}\int_V \delta^2 W_{\mathrm{C}}\,\mathrm{d}V + \cdots. \tag{54}$$

Finally, using the positive-definiteness property of the function $W_{\mathrm{C}}(\tau_{pq}, m_{rpq})$ in (54), we conclude that $\Delta\Pi_{\mathrm{C}} > 0$ and, indeed, that the complementary potential energy assumes a minimum value for the true stress state of the body.

7. A variational principle of the Hellinger–Reissner type

As is well known within the classical theory, instead of a variational principle based on variations of, solely, the displacement or the stress field, one may construct a more general variational principle based on the variation of *both* fields (see, e.g., Fung, 1965; Dym and Shames, 1973). This is the Hellinger–Reissner Principle, which when applied in finite-element procedures gives generally more accurate results than the standard Principles of Virtual Work and Complementary Virtual Work and their associated minimum theorems.

Here, therefore, prompted by the standard Hellinger–Reissner Principle, we consider a functional the extremization of which provides *all* basic equations of the non-linear dipolar gradient elasticity. By observing that there are four independent kinematical variables in the problem, i.e. $(u_q, D(u_q), \varepsilon_{pq}, \kappa_{rpq})$, we introduce the four Lagrange multipliers $(\mu_q, \rho_q, \lambda_{pq}, \nu_{rpq})$ for the variation of the functional. Then, the following potential-energy expression (functional) of the Hellinger–Reissner type is constructed:

$$\begin{aligned} \Pi_R = & \int_V \left[W(\varepsilon_{pq}, \kappa_{rpq}) - f_q u_q - \lambda_{pq}\left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\right]\mathrm{d}V - \int_V \nu_{rpq}(\kappa_{rpq} - \partial_r\partial_p u_q)\mathrm{d}V \\ & - \int_{S_\sigma}\left[P_q^* u_q + R_q^* D(u_q)\right]\mathrm{d}S - \int_{S_u}\mu_q(u_q - u_q^*)\mathrm{d}S - \int_{S_u}\rho_q\left[D(u_q) - D(u_q^*)\right]\mathrm{d}S, \end{aligned} \tag{55}$$

where $(u_q^*, D(u_q^*))$ and (P_q^*, R_q^*) are known values of the corresponding fields on S_u and S_σ, respectively.

To compute the variation of Π_R, the quantities $(u_q, D(u_q), \varepsilon_{pq}, \kappa_{rpq})$ and $(\mu_q, \rho_q, \lambda_{pq}, \nu_{rpq})$ are all treated as *independent*. The variation of the above functional is

$$\begin{aligned} \delta\Pi_R = & \int_V \left[\frac{\partial W}{\partial\varepsilon_{pq}}\delta\varepsilon_{pq} + \frac{\partial W}{\partial\kappa_{rpq}}\delta\kappa_{rpq} - f_q\delta u_q - \lambda_{pq}\delta\varepsilon_{pq} - \varepsilon_{pq}\delta\lambda_{pq} + \lambda_{(pq)}\partial_p(\delta u_q)\right]\mathrm{d}V \\ & + \int_V \left[(\partial_p u_q)\delta\lambda_{(pq)} - \nu_{rpq}\delta\kappa_{rpq} - \kappa_{rpq}\delta\nu_{rpq} + \nu_{(rp)q}\partial_r(\partial_p(\delta u_q)) + \partial_r(\partial_p u_q)\delta\nu_{(rp)q}\right]\mathrm{d}V \\ & - \int_{S_u}\left[P_q^*\delta u_q + R_q^* D(\delta u_q)\right]\mathrm{d}S - \int_{S_u}(u_q - u_q^*)\delta\mu_q\,\mathrm{d}S - \int_{S_u}\mu_q\delta u_q\,\mathrm{d}S \\ & - \int_{S_u}\left[D(u_q) - D(u_q^*)\right]\delta\rho_q\,\mathrm{d}S - \int_{S_u}\rho_q D(\delta u_q)\mathrm{d}S, \end{aligned} \tag{56}$$

where ( ) as a subscript denotes the symmetric part of a tensor. In (56), however, certain terms appear that exhibit non-explicit dependence upon the variations. These are the last term of the first integral and the fourth term of the second integral in the latter expression. Working on these terms, we use the Green–Gauss theorem, the definition of the surface gradient operator, and Eq. (21). After some algebra and employing also the symmetry of the strain tensor, we finally obtain

$$\begin{aligned} \delta\Pi_R = & \int_V \left[\left(\frac{\partial W}{\partial\varepsilon_{pq}} - \lambda_{pq}\right)\delta\varepsilon_{pq} + \left(\frac{\partial W}{\partial\kappa_{spq}} - \nu_{spq}\right)\delta\kappa_{spq} - \left[f_q + \partial_p\lambda_{(pq)} - \partial_p(\partial_r\nu_{(rp)q})\right]\delta u_q\right]\mathrm{d}V \\ & - \int_V \left[\left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\delta\lambda_{pq} + (\kappa_{rpq} - \partial_r\partial_p u_q)\delta\nu_{rpq}\right]\mathrm{d}V - \int_{S_u}(u_q - u_q^*)\delta\mu_q\,\mathrm{d}S \\ & - \int_{S_u}\left[D(u_q) - D(u_q^*)\right]\delta\rho_q\,\mathrm{d}S - \int_{S_\sigma}\left[R_q^* - n_r n_p \nu_{(rp)q}\right]D(\delta u_q)\mathrm{d}S \\ & - \int_{S_\sigma}\left[P_q^* - n_p\lambda_{(pq)} + D_p(n_r\nu_{(rp)q}) - (D_l n_l)n_r n_p \nu_{(rp)q} + n_p\partial_r\nu_{(rp)q}\right]\delta u_q\,\mathrm{d}S \\ & - \int_{S_u}\left[\mu_q - n_p\lambda_{(pq)} + D_p(n_r\nu_{(rp)q}) - (D_l n_l)n_r n_p \nu_{(rp)q} + n_p\partial_r\nu_{(rp)q}\right]\delta u_q\,\mathrm{d}S \\ & - \int_{S_u}\left[\rho_q - n_r n_p \nu_{(rp)q}\right]D(\delta u_q)\mathrm{d}S. \end{aligned} \tag{57}$$

Now, setting $\delta\Pi_R = 0$ one obtains all the basic equations of the non-linear dipolar gradient elasticity (i.e. the definition of stresses, the equations of equilibrium, the definition of strain in terms of displacement gradients, the compatibility relations, and the kinematical and traction boundary conditions) while the Lagrange multipliers are found to correspond to the stresses and the auxiliary tractions, viz.

$$\nu_{rpq} \equiv m_{rpq}, \tag{58a}$$
$$\lambda_{pq} \equiv \tau_{pq}, \tag{58b}$$
$$\mu_q \equiv P_q = n_p(\lambda_{(pq)} - \partial_r \nu_{(rp)q}) - D_p(n_r \nu_{(rp)q}) + (D_l n_l) n_r n_p \nu_{(rp)q}, \tag{58c}$$
$$\rho_q \equiv R_q = n_r n_p \nu_{(rp)q}. \tag{58d}$$

Therefore, in view of the above results, we rewrite (55) under the following form obtaining the analogue of the Hellinger–Reissner potential energy within the present theory:

$$\begin{aligned} \Pi_R = & \int_V \left[W(\varepsilon_{pq}, \kappa_{rpq}) - f_q u_q - \tau_{pq}\left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\right] dV - \int_V m_{rpq}(\kappa_{rpq} - \partial_r \partial_p u_q) dV \\ & - \int_{S_\sigma} \left[P_q^* u_q + R_q^* D(u_q)\right] dS - \int_{S_u} P_q(u_q - u_q^*) dS - \int_{S_u} R_q \left[D(u_q) - D(u_q^*)\right] dS. \end{aligned} \tag{59}$$

8. Theorems of Castigliano and Engesser type

As in the case of classical theory, two useful theorems for structural analysis can be derived within the present theory by invoking the Principles of Virtual Work and Complementary Virtual Work. For a structure in equilibrium under a set of *discrete* conservative tractions (auxiliary) $P_q^{(\alpha)} (\alpha = 1, 2, \ldots, n)$ and $R_q^{(\beta)} (\beta = 1, 2, \ldots, m)$, where $1, 2, \ldots$ are the points of application of the loads, the Principle of Virtual Work in (25a) states that

$$\delta U = \sum_{\alpha=1}^{n} P_q^{(\alpha)} \delta u_q^{(\alpha)} + \sum_{\beta=1}^{m} R_q^{(\beta)} D(\delta u_q^{(\beta)}), \tag{60}$$

where u_q is the displacement at the point of application of the individual force and in the same direction. The points of application of $P_q^{(\alpha)}$ may not coincide with the points of application of $R_q^{(\beta)}$.

Further, compatible with the theory employed here is to assume that the strain energy U is a function of the displacement and its normal derivative. Accordingly, the first variation of U will be written as

$$\delta U = \sum_{\alpha=1}^{n} \frac{\partial U}{\partial u_q^{(\alpha)}} \delta u_q^{(\alpha)} + \sum_{\beta=1}^{m} \frac{\partial U}{\partial D(u_q^{(\beta)})} D(\delta u_q^{(\beta)}). \tag{61}$$

Comparing now (60) and (61), we deduce a theorem of the Castigliano type, viz.

$$\frac{\partial U}{\partial u_q^{(\alpha)}} = P_q^{(\alpha)}, \quad \frac{\partial U}{\partial D(u_q^{(\beta)})} = R_q^{(\beta)}. \tag{62a, b}$$

Another theorem (a theorem of the Engesser type) can be obtained from the Principle of Complementary Virtual Work working along similar lines. Indeed, (46a) provides

$$\delta U_C = \sum_{\alpha=1}^{n} u_q^{(\alpha)} \delta P_q^{(\alpha)} + \sum_{\beta=1}^{m} D(u_q^{(\beta)}) \delta R_q^{(\beta)}, \tag{63}$$

whereas the first variation of the complementary strain energy U_C is written as

$$\delta U_C = \sum_{\alpha=1}^{n} \frac{\partial U_C}{\partial P_q^{(\alpha)}} \delta P_q^{(\alpha)} + \sum_{\beta=1}^{m} \frac{\partial U_C}{\partial R_q^{(\beta)}} \delta R_q^{(\beta)}, \tag{64}$$

assuming—as is natural—that U_C is a function of the single force and double force tractions. Finally, from (63) and (64), we end up with the following result

$$\frac{\partial U_C}{\partial P_q^{(\alpha)}} = u_q^{(\alpha)}, \quad \frac{\partial U_C}{\partial R_q^{(\beta)}} = D(u_q^{(\beta)}). \tag{65a, b}$$

9. Uniqueness theorem of the Kirchhoff–Neumann type

We prove here a theorem of uniqueness within the linear version of the dipolar gradient elasticity theory and under the provision of a positive definite strain-energy density. Following the strategy of Neumann's theorem (in classical elasticity), which is more general than the standard Kirchhoff theorem (dealing only with traction boundary conditions), we consider the case of mixed boundary conditions ((P_q, R_q) given on S_σ and $(u_q, D(u_q))$ given on S_u). The proof proceeds by *reductio ad absurdum*. To this end, we assume that two different solutions do exist for the same problem (i.e. same material, geometry, boundary conditions and body forces), say $\Lambda' = (u'_q, D(u'_q), \tau'_{pq}, m'_{rpq})$ and $\Lambda'' = (u''_q, D(u''_q), \tau''_{pq}, m''_{rpq})$.

Then, due to the assumed identical boundary conditions, we write

$$\int_S \left[(P'_q - P''_q)(u'_q - u''_q) + (R'_q - R''_q)(D(u'_q) - D(u''_q))\right] \mathrm{d}S = 0. \tag{66}$$

In addition, due to the linearity of the governing equations, the difference of the two solutions defined as $\Lambda = (u_q, D(u_q), \tau_{pq}, m_{rpq})$, where $u_q = u'_q - u''_q, D(u_q) = D(u'_q) - D(u''_q), \tau_{pq} = \tau'_{pq} - \tau''_{pq}$ and $m_{rpq} = m'_{rpq} - m''_{rpq}$, will satisfy Eqs. (6) and (8)–(10) with $f_q = f'_q - f''_q \equiv 0, P_q = P'_q - P''_q \equiv 0$ and $R_q = R'_q - R''_q \equiv 0$. Since now the LHS of (25b) (the expression of the Principle of Virtual Work) vanishes, due to (66) and the fact that $f_q \equiv 0$, we are led to the conclusion that the RHS of (25b) vanishes too, which, however, is not true because the strain energy was assumed to be a *positive definite* quantity.

10. The reciprocal theorem

Here, we were able to prove a reciprocal theorem that is analogous to the classical Betti–Rayleigh theorem when the linear constitutive relations have the following form:

$$\tau_{pq} = c_{pqlm}\varepsilon_{lm}, \quad m_{rpq} = d_{rpqjlm}\kappa_{jlm}, \tag{67a, b}$$

i.e. when the tensor of material constants f_{rpqlm} in the general linear relations (6) is *absent*. As mentioned in Section 2, this tensor (being of odd rank) inevitably results in preferred directions in the material response. In the case where f_{rpqlm} does not vanish, a reciprocal theorem cannot be established. However, this restriction is not very serious because Eq. (67) are general enough and still allow for anisotropic material response. This result of ours (i.e. of having f_{rpqlm} vanished in order to establish a reciprocal theorem) was also corroborated by Giannakopoulos et al. (submitted for publication).

Consider two different sets of forces (f_q, P_q, R_q) and (f'_q, P'_q, R'_q) acting successively on the body. The first set (unprimed one) and the second set (primed one) result, respectively, in the following sets of strains and stresses $(\tau_{pq}, m_{rpq}, \varepsilon_{pq}, \kappa_{rpq})$ and $(\tau'_{pq}, m'_{rpq}, \varepsilon'_{pq}, \kappa'_{rpq})$. Then, applying the Principle of Virtual Work yields

$$\int_V f_q u'_q \,\mathrm{d}V + \int_{S_\sigma} \left[P_q u'_q + R_q D(u'_q)\right] \mathrm{d}S = \int_V (\tau_{pq}\varepsilon'_{pq} + m_{rpq}\kappa'_{rpq}) \mathrm{d}V, \tag{68}$$

if we assume that the forces of the unprimed set act first and the field $(u'_q, D(u'_q))$ produced by the primed set of loads is identified with the *virtual* field. If, however, the succession of application of the loads is reversed, the Principle of Virtual Work will provide

$$\int_V f'_q u_q \,\mathrm{d}V + \int_{S_\sigma} \left[P'_q u_q + R'_q D(u_q)\right] \mathrm{d}S = \int_V (\tau'_{pq}\varepsilon_{pq} + m'_{rpq}\kappa_{rpq}) \mathrm{d}V. \tag{69}$$

Now, from the constitutive equations and the symmetries in (5) we have the relations

$$\tau_{pq}\varepsilon'_{pq} = c_{pqlm}\varepsilon_{lm}\varepsilon'_{pq} = c_{lmpq}\varepsilon'_{pq}\varepsilon_{lm} = \tau'_{lm}\varepsilon_{lm} = \tau'_{pq}\varepsilon_{pq}, \tag{70}$$

$$m_{rpq}\kappa'_{rpq} = d_{rpqjlm}\kappa_{jlm}\kappa'_{rpq} = d_{jlmrpq}\kappa'_{rpq}\kappa_{jlm} = m'_{jlm}\kappa_{jlm} = m'_{rpq}\kappa_{rpq}, \tag{71}$$

which lead to the (intermediate) result

$$\int_V (\tau_{pq}\varepsilon'_{pq} + m_{rpq}\kappa'_{rpq}) \mathrm{d}V = \int_V (\tau'_{pq}\varepsilon_{pq} + m'_{rpq}\kappa_{rpq}) \mathrm{d}V. \tag{72}$$

Finally, combining (72) with Eqs. (68) and (69) provides the mathematical expression of the reciprocal theorem

$$\int_V f_q u'_q \,\mathrm{d}V + \int_{S_\sigma} \left[P_q u'_q + R_q D(u'_q)\right] \mathrm{d}S = \int_V f'_q u_q \,\mathrm{d}V + \int_{S_\sigma} \left[P'_q u_q + R'_q D(u_q)\right] \mathrm{d}S, \tag{73}$$

which states that the work done by the first set of loads acting through the kinematical field produced by the second set of loads is equal to the work done by the forces of the second set of loads acting through the kinematical field produced by the first set. Practically speaking, one disposes in this way an *integral equation* to solve for any unknown kinematical field. The latter field and the system of loading (prescribed, of course) that has produced it are interrelated, via (73), with a known solution (perhaps a Green's function).

A restrictive form of the reciprocal theorem dealing with form II of Mindlin's theory (pure strain-gradient case) and with only isotropic material response was given earlier in the M.S. Thesis of the second author (Grentzelou, 2002) and in the work by Polyzos et al. (2003). Also, Giannakopoulos et al. (submitted for publication) presented a reciprocal theorem dealing with form II of Mindlin's theory and valid for anisotropic material behavior as well.

11. A path-independent integral of the *J* type

Within the framework of non-linear dipolar gradient elasticity, we will derive first the expression giving the energy release rate (ERR) resulting from a virtual extension of a straight crack in a two-dimensional (2D) body. Then, we will prove the path independence of this expression. We should notice that such an expression (*J*-integral) was stated before by Chen et al. (1999) and by Georgiadis (2003), but in both cases without providing any actual proof. Here, we therefore intend for a rigorous derivation of the new form of the *J*-integral and a proof of its path independence. In what follows, the absence of body forces is assumed.

Fig. 4 depicts the 2D configuration of the cracked area A under plane-strain conditions. This area is enclosed by a piecewise smooth curve Γ (this contour runs in a counterclockwise sense starting from the lower crack face and ending on the upper crack face). The curve Γ may either coincide with the boundary of the cracked body or be an imagined curve surrounding the crack tip. The initial crack has length a. Also, an Ox_1x_2 Cartesian coordinate system is attached to the crack tip and translates with it when the crack grows. As is well-known in the case of classical elasticity with non-linear constitutive relations, the negative of the rate of change of total potential energy with respect to crack dimension is called the ERR. This quantity is a function of crack size, in general. From its definition, the ERR is the amount of energy (per unit length along the crack edge) supplied by the elastic energy in the body and by the loading in creating the new fracture surfaces. In this sense, the ERR is a *configurational force* that is work-conjugate to the amount of crack advance (see, e.g., Rice, 1968; Eshelby, 1975; Cherepanov, 1979; Eischen and Herrmann, 1987; Maugin, 1993; Anderson, 1995). Within the classical non-linear elasticity, the ERR was found to be equal to the path-independent *J*-integral of Rice.

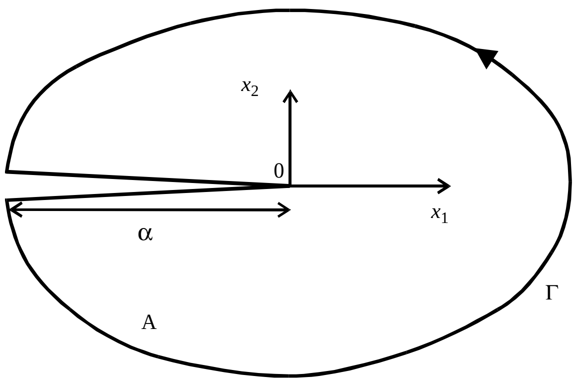

Fig. 4. A two-dimensional cracked area A bounded by the curve Γ.

We work now with the theory of dipolar gradient elasticity involving non-linear constitutive relations. Assuming quasi-static conditions and the absence of body forces, we write from Eq. (26) the potential energy per unit thickness of the body as

$$\Pi = \int_A W(\varepsilon_{pq}, \kappa_{rpq}) \mathrm{d}A - \int_{\Gamma_\sigma} \left[P_q u_q + R_q D(u_q)\right] \mathrm{d}\Gamma, \tag{74}$$

where the first integral is an area integral, the second integral is a line integral, $\mathrm{d}\Gamma$ denotes arc length along the contour Γ, and Γ_σ is the portion of the contour on which tractions are assumed to be prescribed. Further, a virtual extension of the crack along its plane would result in the following change in potential energy

$$\frac{\mathrm{d}\Pi}{\mathrm{d}a} = \int_A \frac{\mathrm{d}W}{\mathrm{d}a} \mathrm{d}A - \int_\Gamma \left[P_q \frac{\mathrm{d}u_q}{\mathrm{d}a} + R_q D\left(\frac{\mathrm{d}u_q}{\mathrm{d}a}\right)\right] \mathrm{d}\Gamma, \tag{75}$$

where the line integration can be extended over the entire contour Γ since $(\mathrm{d}u_q/\mathrm{d}a) = 0$ and $D(\mathrm{d}u_q/\mathrm{d}a) = 0$ over the portion Γ_u, where kinematical conditions are specified. Of course, kinematical conditions can be prescribed along a portion of Γ only if the curve Γ is the boundary of the body. If the curve Γ is simply an imagined curve surrounding the crack tip, then this curve should be identical to Γ_σ.

When the crack grows, the coordinate system moves with the crack tip. We therefore have

$$\frac{\mathrm{d}}{\mathrm{d}a} = \frac{\partial}{\partial a} + \frac{\partial x_1}{\partial a}\frac{\partial}{\partial x_1} = \frac{\partial}{\partial a} - \frac{\partial}{\partial x_1}, \tag{76}$$

since $(\partial x_1/\partial a) = -1$. In view of the above result, Eq. (75) becomes

$$\frac{\mathrm{d}\Pi}{\mathrm{d}a} = \int_A \left(\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x_1}\right) \mathrm{d}A - \int_\Gamma P_q \left(\frac{\partial u_q}{\partial a} - \frac{\partial u_q}{\partial x_1}\right) \mathrm{d}\Gamma - \int_\Gamma R_q \left[D\left(\frac{\partial u_q}{\partial a}\right) - D\left(\frac{\partial u_q}{\partial x_1}\right)\right] \mathrm{d}\Gamma. \tag{77}$$

On the other hand, by invoking the definition of the strain-energy density, i.e. $W \equiv W(\varepsilon_{pq}, \kappa_{rpq})$, we write

$$\frac{\partial W}{\partial a} = \frac{\partial W}{\partial \varepsilon_{pq}}\frac{\partial \varepsilon_{pq}}{\partial a} + \frac{\partial W}{\partial \kappa_{rpq}}\frac{\partial \kappa_{rpq}}{\partial a} = \tau_{pq}\frac{\partial \varepsilon_{pq}}{\partial a} + m_{rpq}\frac{\partial \kappa_{rpq}}{\partial a}, \tag{78}$$

whereas, due to the Principle of Virtual Work (cf. (25)), it is valid

$$\int_A \tau_{pq}\frac{\partial \varepsilon_{pq}}{\partial a} \mathrm{d}A + \int_A m_{rpq}\frac{\partial \kappa_{rpq}}{\partial a} \mathrm{d}A = \int_\Gamma P_q \frac{\partial u_q}{\partial a} \mathrm{d}\Gamma + \int_\Gamma R_q D\left(\frac{\partial u_q}{\partial a}\right) \mathrm{d}\Gamma. \tag{79}$$

Now, in light of (78) and (79), Eq. (77) becomes

$$\frac{\mathrm{d}\Pi}{\mathrm{d}a} = -\int_A \frac{\partial W}{\partial x_1} \mathrm{d}A + \int_\Gamma \left[P_q \frac{\partial u_q}{\partial x_1} + R_q D\left(\frac{\partial u_q}{\partial x_1}\right)\right] \mathrm{d}\Gamma, \tag{80}$$

or, due to the Green–Gauss theorem,

$$J \equiv -\frac{\mathrm{d}\Pi}{\mathrm{d}a} = \int_\Gamma \left[W n_1 - P_q \frac{\partial u_q}{\partial x_1} - R_q D\left(\frac{\partial u_q}{\partial x_1}\right)\right] \mathrm{d}\Gamma = \int_\Gamma \left(W \mathrm{d}x_2 - \left[P_q \frac{\partial u_q}{\partial x_1} + R_q D\left(\frac{\partial u_q}{\partial x_1}\right)\right] \mathrm{d}\Gamma\right), \tag{81}$$

where the symbol "≡" above means equality by definition. The above expression constitutes the new form of J-integral in the non-linear gradient elasticity.

At this point, however, we should more carefully inspect on the applicability of the Green–Gauss theorem in the previous procedure. This is because of the possible presence of singularities of the fields involved at the crack tip. Such singularities would *invalidate* the application of the Green–Gauss theorem (its applicability was taken for granted before). More specifically, we inspect on the possibility of the term $\int_A (\mathrm{d}W/\mathrm{d}a)\mathrm{d}A$ becoming infinite in (75). First, in light of (76), this term is written as

$$\int_A \frac{\mathrm{d}W}{\mathrm{d}a} \mathrm{d}A = \int_A \frac{\partial W}{\partial a} \mathrm{d}A - \int_A \frac{\partial W}{\partial x_1} \mathrm{d}A. \tag{82}$$

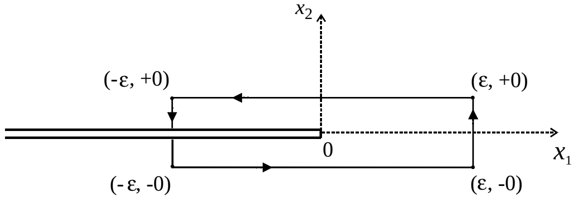

Fig. 5. Rectangular-shaped contour surrounding the crack tip. To evaluate the ERR, $\varepsilon \to +0$ is taken.

Now, by choosing a *rectangular-shaped* contour surrounding the crack tip (see Fig. 5) and by applying the Green–Gauss theorem (accepting for the moment its applicability), the second term in the RHS of (82) is written as

$$\int_A \frac{\partial W}{\partial x_1} \mathrm{d}A = \int_\Gamma W n_1 \, \mathrm{d}\Gamma = \int_\Gamma W \, \mathrm{d}x_2, \tag{83}$$

which, certainly, becomes zero if we allow the 'height' of the rectangle to vanish. We should mention that this convenient concept of the rectangular-shaped contour was introduced by Freund (1972) in examining the energy flux into the tip of a rapidly extending crack. This was also employed by Burridge (1976) and Georgiadis (2003) in different contexts.

In view of the above, we are left in the vicinity of the crack tip with the term $\int_A (\partial W/\partial a)\mathrm{d}A$. Further, due to the definition of W and in view of (79), this term is written as

$$\int_A \frac{\partial W}{\partial a} \mathrm{d}A = \int_A \left(\tau_{pq} \frac{\partial \varepsilon_{pq}}{\partial a} + m_{rpq} \frac{\partial \kappa_{rpq}}{\partial a} \right) \mathrm{d}A = \int_\Gamma \left(P_q \frac{\partial u_q}{\partial a} + R_q D\left(\frac{\partial u_q}{\partial a} \right) \right) \mathrm{d}\Gamma. \tag{84}$$

But, in light of the work by Shi et al. (2000), Georgiadis (2003) and Grentzelou and Georgiadis (2005) on crack problems of linear gradient elasticity, the field $\partial_p u_q$ should be bounded in the crack-tip vicinity (the latter work provides the necessary conditions for solution uniqueness of plane crack problems within the dipolar elasticity theory).

Then, by considering again the rectangular-shaped contour in Fig. 5 and allowing the 'height' of the rectangle to vanish, we end up with line integrals over $-\varepsilon \leqslant x_1 \equiv x \leqslant \varepsilon$ (with $\varepsilon \to +0$). Typically, these integrals have the forms (Georgiadis, 2003): $\int_{-\varepsilon}^{\varepsilon} (x_+)^{-3/2} (x_-)^{1/2} \mathrm{d}x$ and $\int_{-\varepsilon}^{\varepsilon} (x_+)^{-1/2} (x_-)^{-1/2} \mathrm{d}x$, where x_+^ω and x_-^ω are *distributions* defined in the following manner (see, e.g., Lauwerier, 1963; Gelfand and Shilov, 1964). The distribution x_+^ω for $Re(\omega) > -1$ is identified with the function $x_+^\omega = x^\omega$ for $x > 0$ and $x_+^\omega = 0$ for $x < 0$. For other values of the complex parameter ω it is defined by analytic continuation of the functional $\langle x_+^\omega, h \rangle \equiv \int_0^\infty x^\omega h(x) \, \mathrm{d}x$, where $h(x)$ is a test function. In this way, a distribution is obtained for all complex values of ω with the exception of $\omega = -1, -2, -3, \ldots$ In a similar manner, x_-^ω is defined by starting from $x_-^\omega = 0$ for $x > 0$ and $x_-^\omega = |x|^\omega$ for $x < 0$. Finally, Fisher's theorem on the product of distributions (Fisher, 1971; Freund, 1990; Georgiadis, 2003), i.e. the operational relation $(x_+)^{-1-\omega} (x_-)^\omega = -\pi\delta(x)[2\sin(\pi\omega)]^{-1}$ with $\omega \neq -1, -2, -3, \ldots$ and $\delta(x)$ being now the Dirac delta distribution, provides a means to evaluate the above integrals. Indeed, in view of the fundamental property of the Dirac delta distribution that $\int_{-\varepsilon}^{\varepsilon} \delta(x) \, \mathrm{d}x = 1$, the integrals are *bounded* (despite the hyper-singular nature of the near-tip stress). Therefore, it turns out that Eq. (84) will give a *finite* result (in the most singular possible case of *linear* gradient elasticity). For more details on calculations of this type, we refer to the work by Georgiadis (2003).

We will show now that the integral in (81) is path-independent, so that J is indeed the ERR for any choice of curve Γ. The path-independent property is a convenient one since it often permits a *direct* evaluation of J. Following the respective procedure of standard elasticity (see, e.g., Rice, 1968; Anderson, 1995), we begin by evaluating J over a closed curve Γ_0 surrounding a region A_0 of the body (see Fig. 6). It is assumed that this region is simply connected and free of singularities. Then, invoking the traction boundary conditions stated in Eqs. (9) and (10), the J-integral takes the form

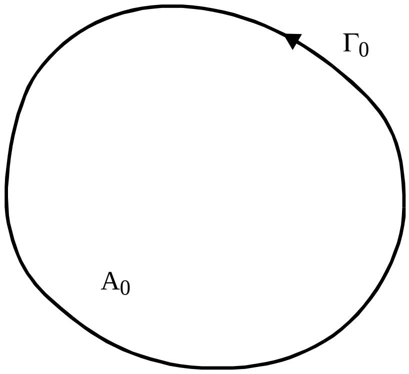

Fig. 6. A closed curve Γ_0 surrounding a region A_0 in a 2D body.

$$\begin{aligned} J &= \int_{\Gamma_0} \left[W n_1 - P_q \frac{\partial u_q}{\partial x_1} - R_q D\left(\frac{\partial u_q}{\partial x_1}\right)\right] \mathrm{d}\Gamma \\ &= \int_{A_0} \left(\frac{\partial W}{\mathrm{d}x_1} - \partial_p \left[(\tau_{pq} - \partial_r m_{rpq}) \frac{\partial u_q}{\partial x_1}\right] - \partial_r \left[n_p m_{rpq} D\left(\frac{\partial u_q}{\partial x_1}\right)\right]\right) \mathrm{d}A \\ &\quad + \int_{\Gamma_0} [D_p(n_r m_{rpq}) - (D_l n_l) n_r n_p m_{rpq}] \frac{\partial u_q}{\partial x_1} \mathrm{d}\Gamma. \end{aligned} \tag{85}$$

Further, working on the first term of the surface integral in (85) and by invoking the definition of the strain-energy density and the symmetries of the tensors involved, we have

$$\frac{\partial W}{\partial x_1} = \frac{\partial W}{\partial \varepsilon_{pq}} \frac{\partial \varepsilon_{pq}}{\partial x_1} + \frac{\partial W}{\partial \kappa_{rpq}} \frac{\partial \kappa_{rpq}}{\partial x_1} = \tau_{pq} \frac{\partial \varepsilon_{pq}}{\partial x_1} + m_{rpq} \frac{\partial \kappa_{rpq}}{\partial x_1} = (\tau_{pq} - \partial_r m_{rpq}) \partial_p \left(\frac{\partial u_q}{\partial x_1}\right) + \partial_r \left[m_{rpq} \partial_p \left(\frac{\partial u_q}{\partial x_1}\right)\right]. \tag{86}$$

Also, by using the result in (23) in conjunction with the Green–Gauss theorem, the line integral in (85) becomes

$$\int_{\Gamma_0} [D_p(n_r m_{rpq}) - (D_l n_l) n_r n_p m_{rpq}] \frac{\partial u_q}{\partial x_1} \mathrm{d}\Gamma = -\int_{\Gamma_0} n_r m_{rpq} D_p \left(\frac{\partial u_q}{\partial x_1}\right) \mathrm{d}\Gamma = -\int_{A_0} \partial_p \left[m_{rpq} D_p \left(\frac{\partial u_q}{\partial x_1}\right)\right] \mathrm{d}A. \tag{87}$$

In view of the above, Eq. (85) takes the form

$$J = -\int_{A_0} \partial_p (\tau_{pq} - \partial_r m_{rpq}) \frac{\partial u_q}{\partial x_1} \mathrm{d}A, \tag{88}$$

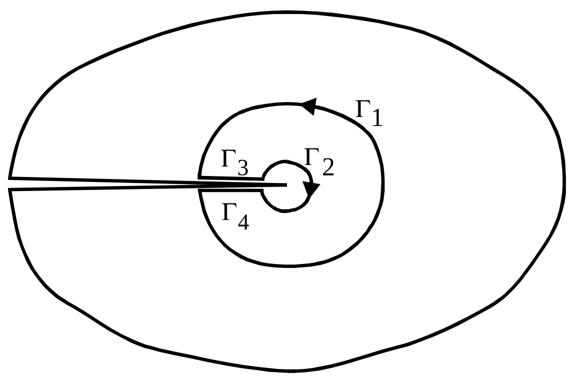

Fig. 7. Two arbitrary contours Γ_1 and Γ_2 surrounding the crack tip. These contours when supplied by the line segments Γ_3 and Γ_4 form a closed contour.

which is equal to zero because the integrand vanishes due to the equilibrium equations (with no body forces) stated in (8). Thus, $J = 0$ for any closed contour.

The final step to prove path-independence is to consider two arbitrary contours Γ_1 and Γ_2 surrounding the crack tip, as shown in Fig. 7. It is assumed that the region between Γ_1 and Γ_2 is simply connected and free of singularities. Then, a *closed* contour can be constructed by connecting the previous contours by the line segments Γ_3 and Γ_4 running, respectively, along the upper and lower crack surfaces. The total J along the closed path (equal to zero, in view of the previous analysis) will be equal to the sum of contributions from each particular contour, i.e. $J = J_1 + J_2 + J_3 + J_4 \equiv 0$. But, along the upper and lower crack surfaces the tractions are vanished and also $dx_2 = 0$. Therefore, $J_3 = J_4 = 0$ from their definition in (81) and $J_1 = -J_2$. Path-independence has thus been proved.

12. Pure strain-gradient case

In the pure strain-gradient case, Eq. (2) for the strain-energy density is considered but now with κ_{rpq} being the gradient of *only* the strain field (and not of the entire displacement-gradient field), i.e. $\kappa_{rpq} = \partial_r \varepsilon_{pq}$. This is form II in Mindlin's (1964) paper. Obviously, it is $\kappa_{rpq} \equiv \kappa_{rqp}$. Stresses are defined as in (3) and, accordingly, the dipolar stress tensor exhibits the latter type of symmetry, i.e. $m_{rpq} \equiv m_{rqp}$. This formulation of the dipolar gradient theory does not take into consideration rotation gradients. All governing equations pertaining to form I (presented in Section 2) and all energy considerations and results given before are also valid for form II provided that the proper symmetries for all tensors are followed. Therefore, in this Section we just present a few intermediate results (valid in the pure strain-gradient case) when deviations occur between the two forms. No derivations are provided.

When linear and isotropic behaviour is considered, (7b) has to be replaced by

$$m_{rpq} = \frac{1}{2} d_1 (\kappa_{qjj}\delta_{rp} + 2\kappa_{jjr}\delta_{pq} + \kappa_{pjj}\delta_{qr}) + 2d_2\kappa_{rjj}\delta_{pq} + d_3(\kappa_{jjq}\delta_{rp} + \kappa_{jjp}\delta_{rq}) + 2d_4\kappa_{rpq} + d_5(\kappa_{pqr} + \kappa_{qrp}). \tag{89}$$

The integral expression in (41) for the Principle of Complementary Virtual Work has to be replaced by

$$\int_V \left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\delta\tau_{pq}\,dV + \int_V \left[\kappa_{rpq} - (1/2)\partial_r(\partial_p u_q + \partial_q u_p)\right]\delta m_{rpq}\,dV$$
$$+ \int_{S_u} (u_q - u_q^*)\delta P_q\,dS + \int_{S_u} \left[D(u_q) - D(u_q^*)\right]\delta R_q\,dS = 0. \tag{90}$$

Finally, Eqs. (55)–(57) and (59) for the Principle of the Hellinger–Reissner type have to be replaced, respectively, by

$$\begin{aligned}\Pi_R = &\int_V \left[W(\varepsilon_{pq}, \kappa_{rpq}) - f_q u_q - \lambda_{pq}\left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\right]dV \\ &- \int_V \nu_{rpq}\left[\kappa_{rpq} - (1/2)\partial_r(\partial_p u_q + \partial_q u_p)\right]dV - \int_{S_\sigma}\left[P_q^* u_q + R_q^* D(u_q)\right]dS \\ &- \int_{S_u} \mu_q (u_q - u_q^*)dS - \int_{S_u} \rho_q\left[D(u_q) - D(u_q^*)\right]dS,\end{aligned} \tag{91}$$

$$\begin{aligned}\delta\Pi_R = &\int_V \left[\frac{\partial W}{\partial \varepsilon_{pq}}\delta\varepsilon_{pq} + \frac{\partial W}{\partial \kappa_{rpq}}\delta\kappa_{rpq} - f_q\delta u_q - \lambda_{pq}\delta\varepsilon_{pq} - \varepsilon_{pq}\delta\lambda_{pq} + \lambda_{(pq)}\partial_p(\delta u_q)\right]dV \\ &+ \int_V \left[(\partial_p u_q)\delta\lambda_{(pq)} - \nu_{rpq}\delta\kappa_{rpq} - \kappa_{rpq}\delta\nu_{rpq} + \nu_{r(pq)}\partial_r(\partial_p(\delta u_q)) + \partial_r(\partial_p u_q)\delta\nu_{r(pq)}\right]dV \\ &- \int_{S_u}\left[P_q^*\delta u_q + R_q^* D(\delta u_q)\right]dS - \int_{S_u}(u_q - u_q^*)\delta\mu_q\,dS - \int_{S_u}\mu_q\delta u_q\,dS \\ &- \int_{S_u}\left[D(u_q) - D(u_q^*)\right]\delta\rho_q\,dS - \int_{S_u}\rho_q D(\delta u_q)dS,\end{aligned} \tag{92}$$

$$\begin{aligned}\delta\Pi_R = &\int_V \left[\left(\frac{\partial W}{\partial \varepsilon_{pq}} - \lambda_{pq}\right)\delta\varepsilon_{pq} + \left(\frac{\partial W}{\partial \kappa_{spq}} - v_{spq}\right)\delta\kappa_{spq} - \left[f_q + \partial_p\lambda_{(pq)} - \partial_p(\partial_r v_{(rp)q})\right]\delta u_q\right]\mathrm{d}V \\ &- \int_V \left[\left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\delta\lambda_{pq} + \left[\kappa_{rpq} - (1/2)\partial_r(\partial_p u_q + \partial_q u_p)\right]\delta v_{rpq}\right]\mathrm{d}V \\ &- \int_{S_u} (u_q - u_q^*)\delta\mu_q\,\mathrm{d}S - \int_{S_u}\left[D(u_q) - D(u_q^*)\right]\delta\rho_q\,\mathrm{d}S - \int_{S_\sigma}\left[R_q^* - n_r n_p v_{r(pq)}\right]D(\delta u_q)\mathrm{d}S \\ &- \int_{S_\sigma}\left[P_q^* - n_p\lambda_{(pq)} + D_p(n_r v_{r(pq)}) - (D_l n_l)n_r n_p v_{r(pq)} + n_p\partial_r v_{r(pq)}\right]\delta u_q\,\mathrm{d}S \\ &- \int_{S_u}\left[\mu_q - n_p\lambda_{(pq)} + D_p(n_r v_{r(pq)}) - (D_l n_l)n_r n_p v_{r(pq)} + n_p\partial_r v_{r(pq)}\right]\delta u_q\,\mathrm{d}S \\ &- \int_{S_u}\left[\rho_q - n_r n_p v_{r(pq)}\right]D(\delta u_q)\mathrm{d}S, \end{aligned} \tag{93}$$

$$\begin{aligned}\Pi_R = &\int_V \left[W(\varepsilon_{pq}, \kappa_{rpq}) - f_q u_q - \tau_{pq}\left[\varepsilon_{pq} - (1/2)(\partial_p u_q + \partial_q u_p)\right]\right]\mathrm{d}V \\ &- \int_V m_{rpq}\left[\kappa_{rpq} - (1/2)\partial_r(\partial_p u_q + \partial_q u_p)\right]\mathrm{d}V - \int_{S_\sigma}\left[P_q^* u_q + R_q^* D(u_q)\right]\mathrm{d}S \\ &- \int_{S_u} P_q(u_q - u_q^*)\mathrm{d}S - \int_{S_u} R_q\left[D(u_q) - D(u_q^*)\right]\mathrm{d}S. \end{aligned} \tag{94}$$

Of course, in all equations of this Section $\kappa_{rpq} = \partial_r\varepsilon_{pq}$ is implied.

13. Concluding remarks

In this work, we derived general energy principles and theorems within the framework of Mindlin's dipolar gradient elasticity. These are the principles of virtual work and complementary virtual work, the theorem of minimum potential energy, the theorem of minimum complementary potential energy, a variational principle analogous to that of the Hellinger–Reissner principle in classical theory, two theorems analogous to those of Castigliano and Engesser in classical theory, a uniqueness theorem of the Kirchhoff–Neumann type, and a reciprocal theorem. These results can be of importance to computational methods for analyzing practical problems. We were also concerned, in the same framework, with a fundamental energetic quantity of fracture mechanics, namely the *J*-integral. This was rigorously derived for two-dimensional cracked bodies. The new form of the *J*-integral was identified with the energy release rate at the tip of a growing crack and its path-independence was proved.

Acknowledgement

This paper is a partial result of the Project PYTHAGORAS II/EPEAEK II (Operational Programme for Educational and Vocational Training II) [Title of the individual program: ''Micro-mechanics of contacts and diffusion of humidity in granular geomaterials'']. This Project is co-funded by the European Social Fund (75%) of the European Union and by National Resources (25%) of the Greek Ministry of Education.

Also, the second author (C.G. Grentzelou) acknowledges with thanks support from the 'Protagoras' program (#65/1400) of NTU Athens and the State Scholarship Foundation of Greece (IKY).

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